On Inequalities for Eigenvalues of Matrices

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ABSTRACT

New inequalities for eigenvalues of matrices are obtained. They make Schur's and Brown's theorem more precise.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

Let $B(\mathbf{C}^n)$ be the set of all linear operators in a Euclidean space $\mathbf{C}^n$. For $A \in B(\mathbf{C}^n)$ we denote $\lambda_k = \lambda_k(A)$ and $s_k = s_k(A)$ $(k = 1, \dots, n)$ the eigenvalues, counting multiplicity, of A and $(AA^*)^{1/2}$, respectively. By $b_k = b_k(A)$ $(k = 1, \dots, n)$ denote the moduli of the eigenvalues of the matrix $A_J = (A - A^*)/2i$: $b_k = |\lambda_k(A_J)|$.

In this paper new inequalities for λ_k, s_k, and b_k are obtained. They make Schur's and also Brown's theorems [1, Chapter 3.1] somewhat more precise.

Below we shall assume $s_1 \geqslant s_2 \geqslant \cdots \geqslant s_n$, $b_1 \geqslant b_2 \geqslant \cdots b_n$.

The aim of this paper is to prove the following theorem.

THEOREM 1. *For any $A \in \mathbf{B}(\mathbf{C}^n)$ the inequalities*

$$\sum_{k=1}^{j} \left(s_{n-k+1}^2 - |\lambda_k|^2 \right) \leqslant 2 \sum_{k=1}^{j} \left(b_k^2 - |\mathrm{Im}\, \lambda_k|^2 \right) \tag{1.1}$$

655 Avenue of the Americas, New York, NY 10010 0024-3795/93/$6.00

and

$$2 \sum_{k=1}^{j} \left(b_{n-k+1}^2 - |\mathrm{Im}\ \lambda_k|^2 \right) \leqslant \sum_{k=1}^{j} \left(s_k^2 - |\lambda_k|^2 \right) \qquad \textit{for all} \quad j = 1, \dots, n \tag{1.2}$$

are valid. For $j = n$ one has the equality

$$|A|_2^2 - \sum_{k=1}^{n} |\lambda_k|^2 = 2\left(|A_J|_2^2 - \sum_{k=1}^{n} |\mathrm{Im}\ \lambda_k|^2 \right). \tag{1.3}$$

Here $|\cdot|_2$ is Schmidt's norm, i.e., $|A|_2^2 = \mathrm{Trace}\ AA^* = \sum_{k=1}^{n} s_k^2$, $|A_J|_2^2 = \sum_{k=1}^{n} b_k^2$. In addition, from out reasonings below it follows that

$$s_1 \geqslant |\lambda_j| \geqslant s_n, \qquad b_1 \geqslant |\mathrm{Im}\ \lambda_j| \geqslant b_n \qquad \text{for all} \quad j = 1, \dots, n. \tag{1.4}$$

Note that the left-hand pair of inequalities in (1.4) is Brown's theorem. The inequality (1.2) makes Schur's theorem more precise.

2. PROOFS OF THEOREM 1 AND ITS COROLLARY

First, we shall prove two lemmata.

Let $\{e_k\}$ be an orthonormal basis of the triangular representation (Schur's basis) of A, i.e.,

$$Ae_k = \sum_{j=1}^{k} a_{jk} e_k, \quad a_{kk} = \lambda_k \qquad (k = 1, \dots, n).$$

From this,

$$A = D + V, \tag{2.1}$$

where

$$Ve_k = \sum_{j=1}^{k-1} a_{jk} e_k, \quad De_k = \lambda_k e_k \qquad (k = 1, \dots, n).$$

That is, D is the diagonal part and V is the nilpotent (upper diagonal) part of A. Denote

$$P_k = \sum_{j=1}^{k} (\cdot, e_j)e_j,$$

where $(\cdot, \cdot)$ is the inner product. It is clear that P_k $(k = 1, \dots, n)$ are projectors onto invariant subspaces of A.

LEMMA 1. *The equalities*

$$|VP_j|_2^2 = |AP_j|_2^2 - \sum_{k=1}^{j} |\lambda_k|^2 = 2|P_j A_J P_j|_2^2 - 2\sum_{k=1}^{j} |\mathrm{Im}\ \lambda_k|^2$$

are true for any $A \in B(\mathbf{C}^n)$ *and* $j = 1, \dots, n$. *Here* $\lambda_1, \lambda_2, \dots, \lambda_j$ *are the eigenvalues of* AP_j, *counting multiplicity.*

Proof of Lemma 1. It is obvious that both matrices V^*D and D^*V are nilpotent. Therefore

$$\mathrm{Trace}\ D^*VP_j = \mathrm{Trace}\ V^*DP_j = 0. \tag{2.2}$$

It is easy to see that

$$\mathrm{Trace}\ D^*DP_j = \sum_{k=1}^{j} |\lambda_k|^2 \qquad \text{for} \quad j = 1, \dots, n.$$

Due to (2.1) and (2.2), we can write

$$|AP|^2 = \mathrm{Trace}\{P_j(D+V)^*(V+D)P_j\} = \mathrm{Trace}(P_jV^*VP + D^*DP_j)$$

$$= |VP_j|_2^2 + \sum_{k=1}^{j} |\lambda_k|^2. \tag{2.3}$$

On the other hand, we have by (2.2)

$$|P_j A_J P_j|_2^2 = \tfrac{1}{4}\mathrm{Trace}\{P_j(A^* - A)^2 P_j\} = \tfrac{1}{4}\mathrm{Trace}\{P_j(D^* + V^* - D - V)^2 P_j\}$$

$$= \tfrac{1}{4}\mathrm{Trace}\{(D^* - D)^2 P_j\} + \tfrac{1}{4}\mathrm{Trace}\{P_j(V^* - V)^2 P_j\}.$$

Set $V_J = (V - V^*)/2i$, $D_J = (D - D^*)/2i$. Therefore

$$|P_j A_J P_j|_2^2 = |D_J P_j|_2^2 + |P_j V_J P_j|_2^2. \tag{2.4}$$

It is clear that

$$|P_j V_J P_j|_2^2 = \frac{1}{2} \sum_{m=1}^{j} \sum_{k=1}^{m-1} |a_{jk}|^2 = \tfrac{1}{2}|VP_j|_2^2.$$

From this and from (2.4) it follows that

$$|VP_j|_2^2 = 2|P_j A_J P_j|_2^2 - 2|P_j D_J P_j|_2^2.$$

Comparing this and (2.3), we arrive at the result because

$$|D_J P_j|_2^2 = \sum_{k=1}^{j} |\mathrm{Im}\ \lambda_k|^2. \qquad \blacksquare$$

Notice that the statement of Lemma 1 for $j = n$ is proved in [3, Chapter 2].

We need the following result [1, Section 4.1.5]. Let M be a nonnegative Hermitian $n \times n$ matrix, and $\{g_k\}$ be an arbitrary orthonormal basis in $\mathbf{C}^n$. Let also $\lambda_1(M) \geqslant \lambda_2(M) \geqslant \cdots \geqslant \lambda_n(M)$ be the eigenvalues of M, counting multiplicity. Then

$$\sum_{k=1}^{j} (Mg_k, g_k) \geqslant \sum_{k=1}^{j} \lambda_{n-k+1}(M) \qquad \text{for all} \quad j = 1, \dots, n. \tag{2.5}$$

LEMMA 2. *Let the dimension of the nullspace of the nilpotent part V of a matrix A be equal to* $r \geqslant 1$. *Then*

$$\sum_{k=1}^{j} s_{n-k+1}^2 \leqslant \sum_{k=1}^{j} |\lambda_k|^2 \leqslant \sum_{k=1}^{j} s_k^2$$

and

$$\sum_{k=1}^{j} b_{n-k+1}^2 \leqslant \sum_{k=1}^{j} |\mathrm{Im}\ \lambda_k|^2 \leqslant \sum_{k=1}^{j} b_k^2 \qquad \text{for all} \quad j = 1, \dots, r.$$

Proof of Lemma 2. We can write

$$VP_j = 0 \qquad \text{for} \quad j = 1, \dots, r, \tag{2.6}$$

where P_j are projectors onto invariant subspaces of V. We have

$$|AP_j|_2^2 = \sum_{k=1}^{j} (A^*Ae_k, e_k) \leqslant \sum_{k=1}^{j} s_k^2(A) \tag{2.7}$$

(see [2, Section 2.4]). According to (2.5) and (2.7), Lemma 1 gives the relations

$$\sum_{k=1}^{j} s_{n-k+1}^2 \leqslant |VP_j|_2^2 + \sum_{k=1}^{j} |\lambda_k|^2 = |AP_j|_2^2 \leqslant \sum_{k=1}^{j} s_k^2. \tag{2.8}$$

Similarly we have by Lemma 1

$$2\sum_{k=1}^{j} b_{n-k+1}^2 \leqslant 2\sum_{k=1}^{j} |\operatorname{Im} \lambda_k|^2 + |VP_j|_2^2 = 2|P_j A_J P_j|_2^2 \leqslant 2\sum_{k=1}^{j} b_k^2. \tag{2.9}$$

Now the result follows from (2.8), (2.9), and (2.10). ∎

Proof of Theorem 1. The relations (2.8) and (2.9) imply

$$\sum_{k=1}^{j} \left(s_{n-k+1}^2 - |\lambda_k|^2\right) \leqslant |VP_j|_2^2 \leqslant 2\sum_{k=1}^{j} \left(b_k^2 - |\operatorname{Im} \lambda_k|^2\right),$$

so (1.1) is proved. The inequality (1.2) is proved similarly. The equality (1.3) follows from Lemma 1 when $j = n$. ∎

Proof of the inequality (1.4). Since the dimension of the nullspace of a nilpotent operator is greater or is equal to 1, we have the result by Lemma 2.

■

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Received 27 November 1991; final manuscript accepted 14 April 1992